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# Fixed Point Theorems and Convergence Theorems for New Nonlinear Operators in Banach Spaces (Banach space theory and related topics)

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# Fixed Point Theorems and Convergence Theorems for New Nonlinear Operators in Banach Spaces

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**Abstract.** Let  $H$  be a real Hilbert space and let  $C$  be a nonempty closed convex subset of  $H$ . A mapping  $T : C \rightarrow H$  is called generalized hybrid if there are  $\alpha, \beta \in \mathbb{R}$  such that

$$\alpha \|Tx - Ty\|^2 + (1 - \alpha) \|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta) \|x - y\|^2$$

for all  $x, y \in C$ . In this article, we extend this class of generalized hybrid mappings in a Hilbert space to more wide classes of nonlinear mappings in a Hilbert space and a Banach space. Then, we prove fixed point theorems and convergence theorems for these classes of nonlinear mappings in a Hilbert space and a Banach space.

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*Key words and phrases.* Banach space, nonlinear operator, fixed point, iteration procedure, equilibrium problem.

## 1 Introduction

Let  $H$  be a real Hilbert space and let  $C$  be a nonempty closed convex subset of  $H$ . Let  $\mathbb{N}$  and  $\mathbb{R}$  be the sets of positive integers and real numbers, respectively. Let  $f : C \times C \rightarrow \mathbb{R}$  be a bifunction. Then, an equilibrium problem (with respect to  $C$ ) is to find  $\hat{x} \in C$  such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C.$$

The set of such solutions  $\hat{x}$  is denoted by  $EP(f)$ , i.e.,

$$EP(f) = \{\hat{x} \in C : f(\hat{x}, y) \geq 0, \quad \forall y \in C\}.$$

For solving the equilibrium problem, let us assume that the bifunction  $f : C \times C \rightarrow \mathbb{R}$  satisfies the following conditions:

- (A1)  $f(x, x) = 0$  for all  $x \in C$ ;
- (A2)  $f$  is monotone, i.e.,  $f(x, y) + f(y, x) \leq 0$  for all  $x, y \in C$ ;
- (A3) for all  $x, y, z \in C$ ,  $\limsup_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y)$ ;
- (A4) for all  $x \in C$ ,  $f(x, \cdot)$  is convex and lower semicontinuous.

The following theorem appears implicitly in Blum and Oettli [3].

**Theorem 1.1.** *Let  $C$  be a nonempty closed convex subset of  $H$  and let  $f$  be a bifunction of  $C \times C$  into  $\mathbb{R}$  satisfying (A1) – (A4). Let  $r > 0$  and  $x \in H$ . Then, there exists  $z \in C$  such that*

$$f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

The following theorem was also given in Combettes and Hirstoaga [8].

**Theorem 1.2.** *Assume that  $f : C \times C \rightarrow \mathbb{R}$  satisfies (A1) – (A4). For  $r > 0$  and  $x \in H$ , define a mapping  $T_r : H \rightarrow C$  as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \right\}$$

for all  $x \in H$ . Then, the following hold:

- (1)  $T_r$  is single-valued;
- (2)  $T_r$  is a firmly nonexpansive mapping, i.e., for all  $x, y \in H$ ,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

- (3)  $F(T_r) = EP(f)$ ;
- (4)  $EP(f)$  is closed and convex.

The following three nonlinear mappings are deduced from a firmly nonexpansive mapping  $T_r$  in a Hilbert space. A mapping  $T : C \rightarrow H$  is said to be nonexpansive, nonspreading [20], and hybrid [32] if

$$\|Tx - Ty\| \leq \|x - y\|,$$

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

and

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all  $x, y \in C$ , respectively. Motivated by these mappings, Aoyama, Iemoto, Kohsaka and Takahashi [1] introduced a class of nonlinear mappings called  $\lambda$ -hybrid in a Hilbert space. Kocourek, Takahashi and Yao [17] also introduced a more wide class of nonlinear mappings containing the class of  $\lambda$ -hybrid mappings: A mapping  $T : C \rightarrow H$  is called generalized hybrid if there are  $\alpha, \beta \in \mathbb{R}$  such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all  $x, y \in C$ . They proved the following fixed point theorem and nonlinear ergodic theorem in a Hilbert space; see Kocourek, Takahashi and Yao [17].

**Theorem 1.3.** *Let  $C$  be a nonempty closed convex subset of a Hilbert space  $H$  and let  $T : C \rightarrow C$  be a generalized hybrid mapping. Then  $T$  has a fixed point in  $C$  if and only if  $\{T^n z\}$  is bounded for some  $z \in C$ .*

**Theorem 1.4.** *Let  $H$  be a Hilbert space and let  $C$  be a closed convex subset of  $H$ . Let  $T : C \rightarrow C$  be a generalized hybrid mapping with  $F(T) \neq \emptyset$  and let  $P$  be the metric projection of  $H$  onto  $F(T)$ . Then, for any  $x \in C$ ,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element  $p$  of  $F(T)$ , where  $p = \lim_{n \rightarrow \infty} PT^n x$ .

In this article, we extend the class of generalized hybrid mappings in a Hilbert space to more wide classes of nonlinear mappings in a Hilbert space and a Banach space. Then, we prove fixed point theorems and convergence theorems for these classes of nonlinear mappings in a Hilbert space and a Banach space.

## 2 Preliminaries

Let  $H$  be a (real) Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\| \cdot \|$ . From [31], we know the following basic equalities. For  $x, y, u, v \in H$  and  $\lambda \in \mathbb{R}$ , we have

$$\| \lambda x + (1 - \lambda)y \|^2 = \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda(1 - \lambda) \|x - y\|^2 \quad (2.1)$$

and

$$2 \langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \quad (2.2)$$

Let  $C$  be a nonempty closed convex subset of  $H$  and  $x \in H$ . Then, we know that there exists a unique nearest point  $z \in C$  such that  $\|x - z\| = \inf_{y \in C} \|x - y\|$ . We denote such a correspondence by  $z = P_C x$ .  $P_C$  is called the metric projection of  $H$  onto  $C$ . It is known that  $P_C$  is nonexpansive and

$$\langle x - P_C x, P_C x - u \rangle \geq 0$$

for all  $x \in H$  and  $u \in C$ ; see [31] for more details.

Let  $E$  be a real Banach space with norm  $\| \cdot \|$  and let  $E^*$  be the dual space of  $E$ . We denote the value of  $y^* \in E^*$  at  $x \in E$  by  $\langle x, y^* \rangle$ . When  $\{x_n\}$  is a sequence in  $E$ , we denote the strong convergence of  $\{x_n\}$  to  $x \in E$  by  $x_n \rightarrow x$  and the weak convergence by  $x_n \rightharpoonup x$ . The modulus  $\delta$  of convexity of  $E$  is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}$$

for every  $\epsilon$  with  $0 \leq \epsilon \leq 2$ . A Banach space  $E$  is said to be uniformly convex if  $\delta(\epsilon) > 0$  for every  $\epsilon > 0$ . A uniformly convex Banach space is strictly convex and reflexive. Let  $C$  be a nonempty closed convex subset of a Banach space  $E$ . A mapping  $T : C \rightarrow E$  is nonexpansive if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in C$ . A mapping  $T : C \rightarrow E$  is quasi-nonexpansive if  $F(T) \neq \emptyset$  and  $\|Tx - y\| \leq \|x - y\|$  for all  $x \in C$  and  $y \in F(T)$ , where  $F(T)$  is the set of fixed points of  $T$ . If  $C$  is a nonempty closed convex subset of a strictly convex Banach space  $E$  and  $T : C \rightarrow C$  is quasi-nonexpansive, then  $F(T)$  is closed and convex; see Itoh and Takahashi [16]. Let  $E$  be a Banach space. The duality mapping  $J$  from  $E$  into  $2^{E^*}$  is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every  $x \in E$ . Let  $U = \{x \in E : \|x\| = 1\}$ . The norm of  $E$  is said to be Gâteaux differentiable if for each  $x, y \in U$ , the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.3)$$

exists. In the case,  $E$  is called smooth. We know that  $E$  is smooth if and only if  $J$  is a single-valued mapping of  $E$  into  $E^*$ . We also know that  $E$  is reflexive if and only if  $J$  is surjective, and  $E$  is strictly convex if and only if  $J$  is one-to-one. Therefore, if  $E$  is a smooth, strictly convex and reflexive Banach space, then  $J$  is a single-valued bijection. The norm of  $E$  is said to be uniformly Gâteaux differentiable if for each  $y \in U$ , the limit (2.3) is attained uniformly for  $x \in U$ . It is also said to be Fréchet differentiable if for each  $x \in U$ , the limit (2.3) is attained uniformly for  $y \in U$ . A Banach space  $E$  is called uniformly smooth if the limit (2.3) is attained uniformly for  $x, y \in U$ . It is known that if the norm of  $E$  is uniformly Gâteaux differentiable, then  $J$  is uniformly norm to weak\* continuous on each bounded subset of  $E$ , and if the norm of  $E$  is Fréchet differentiable, then  $J$  is norm to norm continuous. If  $E$  is uniformly smooth,  $J$  is uniformly norm to norm continuous on each bounded subset of  $E$ . For more details, see [28, 29]. The following results are also in [28, 29].

**Theorem 2.1.** *Let  $E$  be a Banach space and let  $J$  be the duality mapping on  $E$ . Then, for any  $x, y \in E$ ,*

$$\|x\|^2 - \|y\|^2 \geq 2\langle x - y, j \rangle,$$

where  $j \in Jy$ .

**Theorem 2.2.** *Let  $E$  be a smooth Banach space and let  $J$  be the duality mapping on  $E$ . Then,  $\langle x - y, Jx - Jy \rangle \geq 0$  for all  $x, y \in E$ . Further, if  $E$  is strictly convex and  $\langle x - y, Jx - Jy \rangle = 0$ , then  $x = y$ .*

Let  $E$  be a smooth Banach space. The function  $\phi: E \times E \rightarrow (-\infty, \infty)$  is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad (2.4)$$

for  $x, y \in E$ , where  $J$  is the duality mapping of  $E$ . We have from the definition of  $\phi$  that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle \quad (2.5)$$

for all  $x, y, z \in E$ . From  $(\|x\| - \|y\|)^2 \leq \phi(x, y)$  for all  $x, y \in E$ , we can see that  $\phi(x, y) \geq 0$ . Further, we can obtain the following equality:

$$2\langle x - y, Jz - Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \quad (2.6)$$

for  $x, y, z, w \in E$ . If  $E$  is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \iff x = y. \quad (2.7)$$

The following result was proved by Xu [39].

**Theorem 2.3.** *Let  $E$  be a uniformly convex Banach space and let  $r > 0$ . Then there exists a strictly increasing, continuous and convex function  $g: [0, \infty) \rightarrow [0, \infty)$  such that  $g(0) = 0$  and*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all  $x, y \in B_r$  and  $\lambda \in \mathbb{R}$  with  $0 \leq \lambda \leq 1$ , where  $B_r = \{z \in E : \|z\| \leq r\}$ .

Let  $l^\infty$  be the Banach space of bounded sequences with supremum norm. Let  $\mu$  be an element of  $(l^\infty)^*$  (the dual space of  $l^\infty$ ). Then, we denote by  $\mu(f)$  the value of  $\mu$  at  $f = (x_1, x_2, x_3, \dots) \in l^\infty$ . Sometimes, we denote by  $\mu_n(x_n)$  the value  $\mu(f)$ . A linear functional  $\mu$  on  $l^\infty$  is called a mean if  $\mu(e) = \|\mu\| = 1$ , where  $e = (1, 1, 1, \dots)$ . A mean  $\mu$  is called a Banach

limit on  $l^\infty$  if  $\mu_n(x_{n+1}) = \mu_n(x_n)$ . We know that there exists a Banach limit on  $l^\infty$ . If  $\mu$  is a Banach limit on  $l^\infty$ , then for  $f = (x_1, x_2, x_3, \dots) \in l^\infty$ ,

$$\liminf_{n \rightarrow \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \rightarrow \infty} x_n.$$

In particular, if  $f = (x_1, x_2, x_3, \dots) \in l^\infty$  and  $x_n \rightarrow a \in \mathbb{R}$ , then we have  $\mu(f) = \mu_n(x_n) = a$ . For the proof of existence of a Banach limit and its other elementary properties, see [28].

### 3 New Classes of Nonlinear Operators in Hilbert Spaces

Let  $H$  be a Hilbert space and let  $C$  be a nonempty closed convex subset of  $H$ . A mapping  $S : C \rightarrow H$  is called super hybrid [17] if there are  $\alpha, \beta, \gamma \in \mathbb{R}$  such that

$$\begin{aligned} & \alpha \|Sx - Sy\|^2 + (1 - \alpha + \gamma) \|x - Sy\|^2 \\ & \leq (\beta + (\beta - \alpha)\gamma) \|Sx - y\|^2 + (1 - \beta - (\beta - \alpha - 1)\gamma) \|x - y\|^2 \\ & \quad + (\alpha - \beta)\gamma \|x - Sx\|^2 + \gamma \|y - Sy\|^2 \end{aligned} \quad (3.1)$$

for all  $x, y \in C$ . We call such a mapping an  $(\alpha, \beta, \gamma)$ -super hybrid mapping. We notice that an  $(\alpha, \beta, 0)$ -super hybrid mapping is  $(\alpha, \beta)$ -generalized hybrid. So, the class of super hybrid mappings contains the class of generalized hybrid mappings. A super hybrid mapping is not quasi-nonexpansive generally. In fact, let us consider a super hybrid mapping  $S$  with  $\alpha = 1$ ,  $\beta = 0$  and  $\gamma = 1$ . Then, we have

$$\|Sx - Sy\|^2 + \|x - Sy\|^2 \leq -\|Sx - y\|^2 + 3\|x - y\|^2 + \|x - Sx\|^2 + \|y - Sy\|^2$$

for all  $x, y \in C$ . This is equivalent to

$$\|Sx - Sy\|^2 + 2\langle x - y, Sx - Sy \rangle \leq 3\|x - y\|^2$$

for all  $x, y \in C$ . In the case of  $H = \mathbb{R}$ , consider  $Sx = 2 - 2x$  for all  $x \in \mathbb{R}$ . Then,

$$\begin{aligned} & |Sx - Sy|^2 + 2\langle x - y, Sx - Sy \rangle \\ & = |2 - 2x - (2 - 2y)|^2 + 2\langle x - y, 2 - 2x - (2 - 2y) \rangle \\ & = 4|x - y|^2 + 4\langle x - y, y - x \rangle \\ & = 0 \leq 3|x - y|^2 \end{aligned}$$

for all  $x, y \in \mathbb{R}$ . Hence  $S$  is super hybrid and  $F(S) \neq \emptyset$ . However,  $S$  is not quasi-nonexpansive. Furthermore, we have that

$$Tx = \frac{1}{2}Sx + \frac{1}{2}x = \frac{1}{2}(2 - 2x) + \frac{1}{2}x = 1 - \frac{1}{2}x$$

and hence  $T$  is nonexpansive. In general, we have the following theorem for generalized hybrid mappings and supper hybrid mappings; see Takahashi, Yao and Kocourek [38].

**Theorem 3.1.** *Let  $C$  be a nonempty closed convex subset of a Hilbert space  $H$  and let  $\alpha, \beta$  and  $\gamma$  be real numbers with  $\gamma \neq -1$ . Let  $S$  and  $T$  be mappings of  $C$  into  $H$  such that  $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$ . Then,  $S$  is  $(\alpha, \beta, \gamma)$ -super hybrid if and only if  $T$  is  $(\alpha, \beta)$ -generalized hybrid. In this case,  $F(S) = F(T)$ .*

Using Theorems 3.1 and 1.3, we have the following fixed point theorem [17] for super hybrid mappings in a Hilbert space.

**Theorem 3.2.** *Let  $C$  be a nonempty bounded closed convex subset of a Hilbert space  $H$  and let  $\alpha, \beta$  and  $\gamma$  be real numbers with  $\gamma \geq 0$ . Let  $S : C \rightarrow C$  be an  $(\alpha, \beta, \gamma)$ -super hybrid mapping. Then,  $S$  has a fixed point in  $C$ .*

Let  $C$  be a nonempty closed convex subset of a Hilbert space  $H$  and let  $\alpha, \beta$  and  $\gamma$  be real numbers. Then,  $U : C \rightarrow H$  is called an  $(\alpha, \beta, \gamma)$ -extended hybrid mapping [11] if

$$\begin{aligned} & \alpha(1 + \gamma)\|Ux - Uy\|^2 + (1 - \alpha(1 + \gamma))\|x - Uy\|^2 \\ & \leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\ & \quad - (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2 \end{aligned}$$

for all  $x, y \in C$ . We call such a mapping an  $(\alpha, \beta, r)$ -extended hybrid mapping. Putting  $\gamma = \frac{-r}{1+r}$  in (3.1) with  $1 + r > 0$ , we get that for all  $x, y \in C$ ,

$$\begin{aligned} & \alpha\|Sx - Sy\|^2 + (1 - \alpha + \frac{-r}{1+r})\|x - Sy\|^2 \\ & \leq (\beta + (\beta - \alpha)\frac{-r}{1+r})\|Sx - y\|^2 + (1 - \beta - (\beta - \alpha - 1)\frac{-r}{1+r})\|x - y\|^2 \\ & \quad + (\alpha - \beta)\frac{-r}{1+r}\|x - Sx\|^2 + \frac{-r}{1+r}\|y - Sy\|^2. \end{aligned}$$

From  $1 + r > 0$ , we have

$$\begin{aligned} & \alpha(1 + r)\|Sx - Sy\|^2 + (1 + r - \alpha(1 + r) - r)\|x - Sy\|^2 \\ & \leq (\beta(1 + r) - (\beta - \alpha)r)\|Sx - y\|^2 + (1 + r - \beta(1 + r) \\ & \quad + (\beta - \alpha - 1)r)\|x - y\|^2 - (\alpha - \beta)r\|x - Sx\|^2 - r\|y - Sy\|^2 \end{aligned}$$

and hence

$$\begin{aligned} & \alpha(1 + r)\|Sx - Sy\|^2 + (1 - \alpha(1 + r))\|x - Sy\|^2 \\ & \leq (\beta + \alpha r)\|Sx - y\|^2 + (1 - (\beta + \alpha r))\|x - y\|^2 \\ & \quad - (\alpha - \beta)r\|x - Sx\|^2 - r\|y - Sy\|^2. \end{aligned}$$

This implies that  $S$  is extended hybrid. The following theorem is in [11].

**Theorem 3.3.** *Let  $C$  be a nonempty closed convex subset of a Hilbert space  $H$  and let  $\alpha, \beta$  and  $\gamma$  be real numbers with  $\gamma \neq -1$ . Let  $T$  and  $U$  be mappings of  $C$  into  $H$  such that  $U = \frac{1}{1+\gamma}T + \frac{\gamma}{1+\gamma}I$ . Then, for  $1 + \gamma > 0$ ,  $T : C \rightarrow H$  is an  $(\alpha, \beta)$ -generalized hybrid mapping if and only if  $U : C \rightarrow H$  is an  $(\alpha, \beta, \gamma)$ -extended hybrid mapping.*

Using Theorems 3.2 and 3.3, we can prove a fixed point theorem [11] for generalized hybrid nonself-mappings in a Hilbert space.

**Theorem 3.4.** *Let  $C$  be a nonempty bounded closed convex subset of a Hilbert space  $H$  and let  $\alpha$  and  $\beta$  be real numbers. Let  $T$  be an  $(\alpha, \beta)$ -generalized hybrid mapping with  $\alpha - \beta \geq 0$  of  $C$  into  $H$ . Suppose that there exists  $m > 1$  such that for any  $x \in C$ ,  $Tx = x + t(y - x)$  for some  $y \in C$  and  $t \in \mathbb{R}$  with  $1 \leq t \leq m$ . Then,  $T$  has a fixed point in  $C$ .*

## 4 Convergence Theorems in Hilbert Spaces

In this section, using the technique developed by Takahashi [26], we prove a nonlinear ergodic theorem of Baillon's type [2] for super hybrid mappings in a Hilbert space. Before proving it, we need the following lemma [11].

**Lemma 4.1.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $T$  be a generalized hybrid mapping from  $C$  into itself. Suppose that  $\{T^n x\}$  is bounded for some  $x \in C$ . Define  $S_n x = \frac{1}{n} \sum_{k=1}^n T^k x$ . Then,  $\lim_{n \rightarrow \infty} \|S_n x - TS_n x\| = 0$ . In particular, if  $C$  is bounded, then*

$$\lim_{n \rightarrow \infty} \sup_{x \in C} \|S_n x - TS_n x\| = 0.$$

Using Lemma 4.1, we can prove the following nonlinear ergodic theorem [11].

**Theorem 4.2.** *Let  $H$  be a Hilbert space and let  $C$  be a nonempty closed convex subset of  $H$ . Let  $\alpha, \beta$  and  $\gamma$  be real numbers with  $\gamma \geq 0$  and let  $S : C \rightarrow C$  be an  $(\alpha, \beta, \gamma)$ -super hybrid mapping with  $F(S) \neq \emptyset$  and let  $P$  be the metric projection of  $H$  onto  $F(S)$ . Then, for any  $x \in C$ ,*

$$S_n x = \frac{1}{n} \sum_{k=1}^n \left( \frac{1}{1+\gamma} S + \frac{\gamma}{1+\gamma} I \right)^k x$$

*converges weakly to  $z \in F(S)$ , where  $z = \lim_{n \rightarrow \infty} PT^n x$  and  $T = \frac{1}{1+\gamma} S + \frac{\gamma}{1+\gamma} I$ .*

We can also prove the following strong convergence theorems [11] of Halpern's type for super hybrid mappings in a Hilbert space.

**Theorem 4.3.** *Let  $H$  be a Hilbert space and let  $C$  be a nonempty closed convex subset of  $H$ . Let  $\gamma$  be a real number with  $\gamma \neq -1$  and let  $S : C \rightarrow H$  be a mapping such that*

$$\|Sx - Sy\|^2 + 2\gamma \langle x - y, Sx - Sy \rangle \leq (1 + 2\gamma) \|x - y\|^2$$

*for all  $x, y \in C$ . Let  $\{\alpha_n\} \subset [0, 1]$  be a sequence of real numbers such that*

$$\alpha_n \rightarrow 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty.$$

*Suppose  $\{x_n\}$  is a sequence generated by  $x_1 = x \in C$ ,  $u \in C$  and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) P_C \left\{ \frac{1}{1+\gamma} Sx_n + \frac{\gamma}{1+\gamma} x_n \right\}, \quad n \in \mathbb{N}.$$

*If  $F(S) \neq \emptyset$ , then the sequence  $\{x_n\}$  converges strongly to an element  $v$  of  $F(S)$ , where  $v = P_{F(S)} u$  and  $P_{F(S)}$  is the metric projection of  $H$  onto  $F(S)$ .*

**Theorem 4.4.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$  and let  $\alpha, \beta$  and  $\gamma$  be real numbers with  $\gamma \geq 0$ . Let  $S : C \rightarrow C$  be a  $(\alpha, \beta, \gamma)$ -super hybrid mapping with  $F(S) \neq \emptyset$  and let  $P$  be the metric projection of  $H$  onto  $F(S)$ . Suppose  $\{x_n\}$  is a sequence generated by  $x_1 = x \in C$ ,  $u \in C$  and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n, \\ z_n = \frac{1}{n} \sum_{k=1}^n \left( \frac{1}{1+\gamma} S + \frac{\gamma}{1+\gamma} I \right)^k x_n \end{cases}$$



for all  $n \in \mathbb{N}$ , where  $0 \leq \alpha_n \leq 1$ ,  $\alpha_n \rightarrow 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ . Then  $\{x_n\}$  converges strongly to  $Pu$ .

## 5 Fixed Point Theorems in Banach Spaces

Let  $E$  be a real Banach space and let  $C$  be a nonempty closed convex subset of  $E$ . Then, a mapping  $T : C \rightarrow E$  is said to be firmly nonexpansive [6] if

$$\|Tx - Ty\|^2 \leq \langle x - y, j \rangle,$$

for all  $x, y \in C$ , where  $j \in J(Tx - Ty)$ . It is known that the resolvent of an accretive operator in a Banach space is a firmly nonexpansive mapping; see [6] and [7]. Using Theorem 2.1, we have that for any  $x, y \in C$  and  $j \in J(Tx - Ty)$ ,

$$\begin{aligned} \|Tx - Ty\|^2 \leq \langle x - y, j \rangle &\iff 0 \leq 2\langle x - Tx - (y - Ty), j \rangle \\ &\implies 0 \leq \|x - y\|^2 - \|Tx - Ty\|^2 \\ &\iff \|Tx - Ty\|^2 \leq \|x - y\|^2 \\ &\iff \|Tx - Ty\| \leq \|x - y\|. \end{aligned}$$

This implies that  $T$  is nonexpansive. We also have that for any  $x, y \in C$  and  $j \in J(Tx - Ty)$ ,

$$\begin{aligned} \|Tx - Ty\|^2 \leq \langle x - y, j \rangle &\iff 0 \leq 2\langle x - Tx - (y - Ty), j \rangle \\ &\iff 0 \leq 2\langle x - Tx, j \rangle + 2\langle Ty - y, j \rangle \\ &\implies 0 \leq \|x - Ty\|^2 - \|Tx - Ty\|^2 + \|Tx - y\|^2 - \|Tx - Ty\|^2 \\ &\iff 0 \leq \|x - Ty\|^2 + \|y - Tx\|^2 - 2\|Tx - Ty\|^2 \\ &\iff 2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2. \end{aligned}$$

This implies that  $T$  is a nonspreading mapping in the sense of norm. Furthermore we have that for any  $x, y \in C$  and  $j \in J(Tx - Ty)$ ,

$$\begin{aligned} \|Tx - Ty\|^2 \leq \langle x - y, j \rangle &\iff 0 \leq 4\langle x - Tx - (y - Ty), j \rangle \\ &\iff 0 \leq 2\langle x - Tx - (y - Ty), j \rangle + 2\langle x - Tx - (y - Ty), j \rangle \\ &\implies 0 \leq \|x - y\|^2 - \|Tx - Ty\|^2 + \|x - Ty\|^2 + \|y - Tx\|^2 - 2\|Tx - Ty\|^2 \\ &\iff 3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|x - Ty\|^2 + \|y - Tx\|^2. \end{aligned}$$

This implies that  $T$  is a hybrid mapping in the sense of norm. Thus, it is natural that we extend a generalized hybrid mapping in a Hilbert space by Kocourek, Takahashi and Yao [17] to Banach spaces as follows: Let  $E$  be a Banach space and let  $C$  be a nonempty closed convex subset of  $E$ . A mapping  $T : C \rightarrow E$  is called generalized hybrid [13] if there are  $\alpha, \beta \in \mathbb{R}$  such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \quad (5.1)$$

for all  $x, y \in C$ . We may also call such a mapping an  $(\alpha, \beta)$ -generalized hybrid mapping. We note that an  $(\alpha, \beta)$ -generalized hybrid mapping is nonexpansive for  $\alpha = 1$  and  $\beta = 0$ ,

nonspreading for  $\alpha = 2$  and  $\beta = 1$ , and hybrid for  $\alpha = \frac{3}{2}$  and  $\beta = \frac{1}{2}$ . We first prove a fixed point theorem for generalized hybrid mappings in a Banach space. For proving this, we need the following lemma; see, for instance, [33] and [28].

**Lemma 5.1.** *Let  $C$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$ , let  $\{x_n\}$  be a bounded sequence in  $E$  and let  $\mu$  be a mean on  $l^\infty$ . If  $g : E \rightarrow \mathbb{R}$  is defined by*

$$g(z) = \mu_n \|x_n - z\|^2, \quad \forall z \in E,$$

*then there exists a unique  $z_0 \in C$  such that*

$$g(z_0) = \min\{g(z) : z \in C\}.$$

Using Lemma 5.1, we can prove the following theorem [13].

**Theorem 5.2.** *Let  $C$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$  and let  $T$  be a mapping of  $C$  into itself. Let  $\{x_n\}$  be a bounded sequence of  $E$  and let  $\mu$  be a mean on  $l^\infty$ . If*

$$\mu_n \|x_n - Ty\|^2 \leq \mu_n \|x_n - y\|^2$$

*for all  $y \in C$ , then  $T$  has a fixed point in  $C$ .*

Using Theorem 5.2 and properties of Banach limit, we prove a fixed point theorem [13] for generalized hybrid mappings in a Banach space.

**Theorem 5.3.** *Let  $E$  be a uniformly convex Banach space and let  $C$  be a nonempty closed convex subset of  $E$ . Let  $T : C \rightarrow C$  be a generalized hybrid mapping. Then the following are equivalent:*

- (a)  $F(T) \neq \emptyset$ ;
- (b)  $\{T^n x\}$  is bounded for some  $x \in C$ .

On the other hand, Kocourek, Takahashi and Yao [18] extended a generalized hybrid mapping in a Hilbert space to Banach spaces as follows: Let  $E$  be a smooth Banach space and let  $C$  be a nonempty closed convex subset of  $E$ . A mapping  $T : C \rightarrow E$  is called generalized nonspreading [18] if there are  $\alpha, \beta, \gamma, \delta \in \mathbb{R}$  such that

$$\begin{aligned} & \alpha\phi(Tx, Ty) + (1 - \alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ & \leq \beta\phi(Tx, y) + (1 - \beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\} \end{aligned} \quad (5.2)$$

for all  $x, y \in C$ , where  $\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$  for  $x, y \in E$ . We call such a mapping an  $(\alpha, \beta, \gamma, \delta)$ -generalized nonspreading mapping. If  $E$  is a Hilbert space, then  $\phi(x, y) = \|x - y\|^2$  for  $x, y \in E$ . So, we obtain the following:

$$\begin{aligned} & \alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 + \gamma\{\|Tx - Ty\|^2 - \|x - Ty\|^2\} \\ & \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 + \delta\{\|Tx - y\|^2 - \|x - y\|^2\} \end{aligned}$$

for all  $x, y \in C$ . This implies that

$$\begin{aligned} & (\alpha + \gamma)\|Tx - Ty\|^2 + \{1 - (\alpha + \gamma)\}\|x - Ty\|^2 \\ & \leq (\beta + \delta)\|Tx - y\|^2 + \{1 - (\beta + \delta)\}\|x - y\|^2 \end{aligned}$$

for all  $x, y \in C$ . That is,  $T$  is a generalized hybrid mapping in a Hilbert space. The following is Kocourek, Takahashi and Yao's fixed point theorem [18].

**Theorem 5.4.** *Let  $E$  be a smooth, strictly convex and reflexive Banach space and let  $C$  be a nonempty closed convex subset of  $E$ . Let  $T$  be a generalized nonspreading mapping of  $C$  into itself. Then, the following are equivalent:*

- (a)  $F(T) \neq \emptyset$ ;
- (b)  $\{T^n x\}$  is bounded for some  $x \in C$ .

Let  $E$  be a smooth, strictly convex and reflexive Banach space and let  $C$  be a nonempty subset of  $E$ . Let  $T$  be a mapping of  $C$  into itself. Define a mapping  $T^*$  as follows:

$$T^*x^* = JTJ^{-1}x^*, \quad \forall x^* \in JC,$$

where  $J$  is the duality mapping on  $E$  and  $J^{-1}$  is the duality mapping on  $E^*$ . A mapping  $T^*$  is called the duality mapping of  $T$ ; see [37] and [12]. It is easy to show that  $T^*$  is a mapping of  $JC$  into itself. In fact, for  $x^* \in JC$ , we have  $J^{-1}x^* \in C$  and hence  $TJ^{-1}x^* \in C$ . So, we have

$$T^*x^* = JTJ^{-1}x^* \in JC.$$

Then,  $T^*$  is a mapping of  $JC$  into itself. Furthermore, we define the duality mapping  $T^{**}$  of  $T^*$  as follows:

$$T^{**}x = J^{-1}T^*Jx, \quad \forall x \in C.$$

It is easy to show that  $T^{**}$  is a mapping of  $C$  into itself. In fact, for  $x \in C$ , we have

$$T^{**}x = J^{-1}T^*Jx = J^{-1}JTJ^{-1}Jx = Tx \in C.$$

So,  $T^{**}$  is a mapping of  $C$  into itself. We know the following result in a Banach space; see [9] and [37].

**Lemma 5.5.** *Let  $E$  be a smooth, strictly convex and reflexive Banach space and let  $C$  be a nonempty subset of  $E$ . Let  $T$  be a mapping of  $C$  into itself and let  $T^*$  be the duality mapping of  $JC$  into itself. Then, the following hold:*

- (1)  $JF(T) = F(T^*)$ ;
- (2)  $\|T^n x\| = \|(T^*)^n Jx\|$  for each  $x \in C$  and  $n \in \mathbb{N}$ .

Let  $E$  be a smooth Banach space, let  $J$  be the duality mapping from  $E$  into  $E^*$  and let  $C$  be a nonempty subset of  $E$ . A mapping  $T : C \rightarrow E$  is called skew-generalized nonspreading if there are  $\alpha, \beta, \gamma, \delta \in \mathbb{R}$  such that

$$\begin{aligned} & \alpha\phi(Ty, Tx) + (1 - \alpha)\phi(Ty, x) + \gamma\{\phi(Tx, Ty) - \phi(x, Ty)\} \\ & \leq \beta\phi(y, Tx) + (1 - \beta)\phi(y, x) + \delta\{\phi(Tx, y) - \phi(x, y)\} \end{aligned} \quad (5.3)$$

for all  $x, y \in C$ , where  $\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$  for  $x, y \in E$ . We call such a mapping an  $(\alpha, \beta, \gamma, \delta)$ -skew-generalized nonspreading mapping. Let  $T$  be an  $(\alpha, \beta, \gamma, \delta)$ -skew-generalized nonspreading mapping. Observe that if  $F(T) \neq \emptyset$ , then  $\phi(Ty, u) \leq \phi(y, u)$  for all  $u \in F(T)$  and  $y \in C$ . Indeed, putting  $x = u \in F(T)$  in (5.3), we obtain

$$\phi(Ty, u) + \gamma\{\phi(u, Ty) - \phi(u, Ty)\} \leq \phi(y, u) + \delta\{\phi(u, y) - \phi(u, y)\}.$$

So, we have that

$$\phi(Ty, u) \leq \phi(y, u) \quad (5.4)$$

for all  $u \in F(T)$  and  $y \in C$ . Now, we can prove a fixed point theorem [13] for skew-generalized nonspreading mappings in a Banach space.

**Theorem 5.6.** *Let  $E$  be a smooth, strictly convex and reflexive Banach space and let  $C$  be a nonempty closed subset of  $E$  such that  $JC$  is closed and convex. Let  $T$  be a skew-generalized nonspreading mapping of  $C$  into itself. Then, the following are equivalent:*

- (a)  $F(T) \neq \emptyset$ ;
- (b)  $\{T^n x\}$  is bounded for some  $x \in C$ .

## 6 Convergence Theorems in Banach Spaces

Let  $E$  be a smooth Banach space and let  $C$  be a nonempty closed convex subset of  $E$ . Let  $T : C \rightarrow E$  be a generalized nonspreading mapping. Then, we have that for any  $u \in F(T)$  and  $x \in C$ ,  $\phi(u, Tx) \leq \phi(u, x)$ . This property can be revealed by putting  $x = u \in F(T)$  in (5.2). Similarly, putting  $y = u \in F(T)$  in (5.2), we obtain that for  $x \in C$ ,

$$\begin{aligned} \alpha\phi(Tx, u) + (1 - \alpha)\phi(x, u) + \gamma\{\phi(u, Tx) - \phi(u, x)\} \\ \leq \beta\phi(Tx, u) + (1 - \beta)\phi(x, u) + \delta\{\phi(u, Tx) - \phi(u, x)\} \end{aligned}$$

and hence

$$(\alpha - \beta)\{\phi(Tx, u) - \phi(x, u)\} + (\gamma - \delta)\{\phi(u, Tx) - \phi(u, x)\} \leq 0. \quad (6.1)$$

Therefore, we have that  $\alpha > \beta$  together with  $\gamma \leq \delta$  implies that

$$\phi(Tx, u) \leq \phi(x, u).$$

Now, we can prove the following nonlinear ergodic theorem [18] for generalized nonspreading mappings in a Banach space.

**Theorem 6.1.** *Let  $E$  be a uniformly convex Banach space with a Fréchet differentiable norm and let  $C$  be a nonempty closed convex sunny generalized nonexpansive retract of  $E$ . Let  $T : C \rightarrow C$  be a generalized nonspreading mapping with  $F(T) \neq \emptyset$  such that  $\phi(Tx, u) \leq \phi(x, u)$  for all  $x \in C$  and  $u \in F(T)$ . Let  $R$  be the sunny generalized nonexpansive retraction of  $E$  onto  $F(T)$ . Then, for any  $x \in C$ ,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

*converges weakly to an element  $q$  of  $F(T)$ , where  $q = \lim_{n \rightarrow \infty} RT^n x$ .*

Using Theorem 6.1, we obtain the following theorem.

**Theorem 6.2.** *Let  $E$  be a uniformly convex Banach space with a Fréchet differentiable norm. Let  $T : E \rightarrow E$  be an  $(\alpha, \beta, \gamma, \delta)$ -generalized nonspreading mapping such that  $\alpha > \beta$  and  $\gamma \leq \delta$ . Assume that  $F(T) \neq \emptyset$  and let  $R$  be the sunny generalized nonexpansive retraction of  $E$  onto  $F(T)$ . Then, for any  $x \in E$ ,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

*converges weakly to an element  $q$  of  $F(T)$ , where  $q = \lim_{n \rightarrow \infty} RT^n x$ .*

Using Theorem 6.1, we can also prove Kocourek, Takahashi and Yao's nonlinear ergodic theorem (Theorem 1.4) in Introduction.

**Remark** We do not know whether a nonlinear ergodic theorem of Baillon's type for non-spreading mappings holds or not.

Next, we prove a weak convergence theorem of Mann's iteration [21] for generalized non-spreading mappings in a Banach space. For proving it, we need the following lemma obtained by Takahashi and Yao [36].

**Lemma 6.3.** *Let  $E$  be a smooth and uniformly convex Banach space and let  $C$  be a nonempty closed subset of  $E$  such that  $J_C$  is closed and convex. Let  $T : C \rightarrow C$  be a generalized nonexpansive mapping such that  $F(T) \neq \emptyset$ . Let  $\{\alpha_n\}$  be a sequence of real numbers such that  $0 \leq \alpha_n < 1$  and let  $\{x_n\}$  be a sequence in  $C$  generated by  $x_1 = x \in C$  and*

$$x_{n+1} = R_C(\alpha_n x_n + (1 - \alpha_n)Tx_n), \quad \forall n \in \mathbb{N},$$

*where  $R_C$  is a sunny generalized nonexpansive retraction of  $E$  onto  $C$ . Then  $\{R_{F(T)}x_n\}$  converges strongly to an element  $z$  of  $F(T)$ , where  $R_{F(T)}$  is a sunny generalized nonexpansive retraction of  $C$  onto  $F(T)$ .*

Using Lemma 6.3 and the technique developed by [14], we can prove the following weak convergence theorem.

**Theorem 6.4.** *Let  $E$  be a uniformly convex and uniformly smooth Banach space and let  $C$  be a nonempty closed convex sunny generalized nonexpansive retract of  $E$ . Let  $T : C \rightarrow C$  be a generalized nonspreading mapping with  $F(T) \neq \emptyset$  such that  $\phi(Tx, u) \leq \phi(x, u)$  for all  $x \in C$  and  $u \in F(T)$ . Let  $R$  be the sunny generalized nonexpansive retraction of  $E$  onto  $F(T)$ . Let  $\{\alpha_n\}$  be a sequence of real numbers such that  $0 \leq \alpha_n < 1$  and  $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ . Then, a sequence  $\{x_n\}$  generated by  $x_1 = x \in C$  and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N}$$

*converges weakly to  $z \in F(T)$ , where  $z = \lim_{n \rightarrow \infty} Rx_n$ .*

Using Theorem 6.4, we can prove the following theorems.

**Theorem 6.5.** *Let  $E$  be a uniformly convex and uniformly smooth Banach space. Let  $T : E \rightarrow E$  be an  $(\alpha, \beta, \gamma, \delta)$ -generalized nonspreading mapping such that  $\alpha > \beta$  and  $\gamma \leq \delta$ . Assume that  $F(T) \neq \emptyset$  and let  $R$  be the sunny generalized nonexpansive retraction of  $E$  onto  $F(T)$ . Let  $\{\alpha_n\}$  be a sequence of real numbers such that  $0 \leq \alpha_n < 1$  and  $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ . Then, a sequence  $\{x_n\}$  generated by  $x_1 = x \in C$  and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N}$$

*converges weakly to  $z \in F(T)$ , where  $z = \lim_{n \rightarrow \infty} Rx_n$ .*

**Theorem 6.6** (Kocourek, Takahashi and Yao [17]). *Let  $H$  be a Hilbert space and let  $C$  be a nonempty closed convex subset of  $H$ . Let  $T : C \rightarrow C$  be a generalized hybrid mapping with  $F(T) \neq \emptyset$  and let  $P$  be the metric projection of  $H$  onto  $F(T)$ . Let  $\{\alpha_n\}$  be a sequence of real numbers such that  $0 \leq \alpha_n < 1$  and  $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ . Then, a sequence  $\{x_n\}$  generated by  $x_1 = x \in C$  and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N}$$

*converges weakly to  $z \in F(T)$ , where  $z = \lim_{n \rightarrow \infty} Px_n$ .*

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